Published by Institute of Physics Publishing for SISSA

Received: *February 20, 2008*
Revised: *April 29, 2008*
Accepted: *May 12, 2008*
Published: *May 27, 2008*

Direction-dependent CMB power spectrum and statistical anisotropy from noncommutative geometry

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Abstract: Modern cosmology has now emerged as a testing ground for theories beyond the standard model of particle physics. In this paper, we consider quantum fluctuations of the inflaton scalar field on certain noncommutative spacetimes and look for noncommutative corrections in the cosmic microwave background (CMB) radiation. Inhomogeneities in the distribution of large scale structure and anisotropies in the CMB radiation can carry traces of noncommutativity of the early universe. We show that its power spectrum becomes direction-dependent when spacetime is noncommutative. (The effects due to noncommutativity can be observed experimentally in the distribution of large scale structure of matter as well.) Furthermore, we have shown that the probability distribution determining the temperature fluctuations is not Gaussian for noncommutative spacetimes.

Keywords: Non-Commutative Geometry, Quantum Groups, Cosmology of Theories beyond the SM.

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1. Introduction

The CMB radiation shows how the universe was like when it was only $400,000$ years old. If photons and baryons were in equilibrium before they decoupled from each other, then the CMB radiation we observe today should have a black body spectrum indicating a smooth early universe. But in 1992, the Cosmic Background Explorer (COBE) satellite detected anisotropies in the CMB radiation, which led to the conclusion that the early universe was not smooth: There were small perturbations in the photon-baryon fluid.

The theory of inflation was introduced [1 − 3] to resolve the fine tuning problems associated with the standard Big Bang cosmology. An important property of inflation is that it can generate irregularities in the universe, which may lead to the formation of structure. Inflation is assumed to be driven by a classical scalar field that accelerates the observed universe towards a perfect homogeneous state. But we live in a quantum world where perfect homogeneity is never attained. The classical scalar field has quantum fluctuations around it and these fluctuations act as seeds for the primordial perturbations over the smooth universe. Thus according to these ideas, the early universe had inhomogeneities and we observe them today in the distribution of large scale structure and anisotropies in the CMB radiation.

Physics at Planck scale could be radically different. It is the regime of string theory and quantum gravity. Inflation stretches a region of Planck size into cosmological scales. So, at the end of inflation, physics at Planck region should leave its signature on the cosmological scales too.

There are indications both from quantum gravity and string theory that spacetime is noncommutative with a length scale of the order of Planck length. In this paper we explore the consequences of such noncommutativity for CMB radiation in the light of recent developments in the field of noncommutative quantum field theories relating to deformed Poincaré symmetry.

The early universe and CMB in the noncommutative framework have been addressed in many places [4 − 11]. In [4], the noncommutative parameter $\theta_{\mu\nu} = -\theta_{\nu\mu} =$ constants with $\theta_{0i} = 0$, ($\mu, \nu = 0, 1, 2, 3$, with 0 denoting time direction), characterizing the Moyal plane is scale dependent, while [6, 8, 7] have considered noncommutativity based on stringy spacetime uncertainty relations. Our approach differs from these authors since our quantum fields obey twisted statistics, as implied by the deformed Poincaré symmetry in quantum theories.

We organize the paper as follows: In section 2, we discuss how noncommutativity breaks the usual Lorentz invariance and indicate how this breaking can be interpreted as invariance under a deformed Poincaré symmetry. In section 3, we write down an expression for a scalar quantum field in the noncommutative framework and show how its two-point function is modified. We review the theory of cosmological perturbations and (direction-independent) power spectrum for $\theta_{\mu\nu} = 0$ in section 4. In section 5, we derive the power spectrum for the noncommutative Groenewold-Moyal plane $\mathcal{A}_\theta$ and show that it is direction-dependent and breaks statistical isotropy. In section 6, we compute the angular correlations using this power spectrum and show that there are nontrivial $\mathcal{O}(\theta^2)$ corrections to the CMB temperature fluctuations. Next, in section 7, we discuss the modifications of the n-point functions for any n brought about by a non-zero $\theta^{\mu\nu}$ and show in particular that the underlying probability distribution is not Gaussian. The paper concludes with section 8.

2. Noncommutative spacetime and deformed Poincaré symmetry

At energy scales close to the Planck scale, the quantum nature of spacetime is expected to become important. Arguments based on Heisenberg's uncertainty principle and Einstein's theory of classical gravity suggest that spacetime has a noncommutative structure at such length scales [12]. We can model such spacetime noncommutativity by the commutation relations [13 − 16]

$$[\widehat{x}_\mu, \widehat{x}_\nu] = i\theta_{\mu\nu} \tag{2.1}$$

where $\theta_{\mu\nu} = -\theta_{\nu\mu}$ are constants and $\widehat{x}_\mu$ are the coordinate functions of the chosen coordinate system:

$$\widehat{x}_\mu(x) = x_\mu. \tag{2.2}$$

The above relations depend on choice of coordinates. The commutation relations given in eq. (2.1) only hold in special coordinate systems and will look quite complicated in other coordinate systems. Therefore, it is important to know that in which coordinate system

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the above simple form for the commutation relations holds. For cosmological applications, it is natural to assume that eq. (2.1) holds in a comoving frame, the coordinates in which galaxies are freely falling. Not only does it make the analysis and comparison with the observation easier, but also make the time coordinate the proper time for us (neglecting the small local accelerations).

The relations (2.1) are not invariant under naive Lorentz transformations either. But they are invariant under a deformed Lorentz Symmetry [17], in which the coproduct on the Lorentz group is deformed while the group structure is kept intact, as we briefly explain below.

The Lie algebra $\mathcal{P}$ of the Poincaré group has generators (basis) $M_{\alpha\beta}$ and P_μ. The subalgebra of infinitesimal generators P_μ is abelian and we can make use of this fact to construct a twist element $\mathcal{F}_\theta$ of the underlying quantum group theory [18 − 20]. Using this twist element, the coproduct of the universal enveloping algebra $\mathcal{U}(\mathcal{P})$ of the Poincaré algebra can be deformed in such a way that it is compatible with the above commutation relations.

The coproduct Δ_0 appropriate for $\theta_{\mu\nu} = 0$ is a symmetric map from $\mathcal{U}(\mathcal{P})$ to $\mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P})$. It defines the action of $\mathcal{P}$ on the tensor product of representations. In the case of the generators X of $\mathcal{P}$, this standard coproduct is

$$\Delta_0(X) = 1 \otimes X + X \otimes 1. \tag{2.3}$$

The twist element is

$$\mathcal{F}_\theta = \exp\left(-\frac{i}{2}\theta^{\alpha\beta} P_\alpha \otimes P_\beta\right), \quad P_\alpha = -i\partial_\alpha. \tag{2.4}$$

(The Minkowski metric with signature $(-,+,+,+)$ is used to raise and lower the indices.)

In the presence of the twist, the coproduct Δ_0 is modified to Δ_θ where

$$\Delta_\theta = \mathcal{F}_\theta^{-1} \Delta_0 \mathcal{F}_\theta. \tag{2.5}$$

It is easy to see that the coproduct for translation generators are not deformed,

$$\Delta_\theta(P_\alpha) = \Delta_0(P_\alpha) \tag{2.6}$$

while the coproduct for Lorentz generators are deformed:

$$\begin{aligned}\Delta_\theta(M_{\mu\nu}) &= 1 \otimes M_{\mu\nu} + M_{\mu\nu} \otimes 1 - \frac{1}{2}\Big[(P\cdot\theta)_\mu \otimes P_\nu - P_\nu \otimes (P\cdot\theta)_\mu - (\mu \leftrightarrow \nu)\Big], \\ (P\cdot\theta)_\lambda &= P_\rho \theta^\rho_\lambda.\end{aligned} \tag{2.7}$$

The algebra $\mathcal{A}_0$ of functions on the Minkowski space $\mathcal{M}^4$ is commutative with the commutative multiplication m_0:

$$m_0(f \otimes g)(x) = f(x)g(x). \tag{2.8}$$

The Poincaré algebra acts on $\mathcal{A}_0$ in a well-known way

$$P_\mu f(x) = -i\partial_\mu f(x), \quad M_{\mu\nu}\, f(x) = -i(x_\mu\partial_\nu - x_\nu\partial_\mu) f(x). \tag{2.9}$$

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It acts on tensor products $f \otimes g$ using the coproduct $\Delta_0(X)$.

This commutative multiplication is changed in the Groenewold-Moyal algebra $\mathcal{A}_\theta$ to m_θ:

$$m_\theta(f \otimes g)(x) = m_0\left[\mathrm{e}^{-\frac{i}{2}\theta^{\alpha\beta}P_\alpha \otimes P_\beta}\, f \otimes g\right](x) = (f \star g)(x). \tag{2.10}$$

Equation (2.1) is a consequence of this $\star$-multiplication:

$$[\widehat{x}_\mu, \widehat{x}_\nu]_\star = m_\theta\left(\widehat{x}_\mu \otimes \widehat{x}_\nu - \widehat{x}_\nu \otimes \widehat{x}_\mu\right) = i\theta_{\mu\nu}. \tag{2.11}$$

The Poincaré algebra acts on functions $f \in \mathcal{A}_\theta$ in the usual way while it acts on tensor products $f \otimes g \in \mathcal{A}_\theta \otimes \mathcal{A}_\theta$ using the coproduct $\Delta_\theta(X)$ [17, 21].

Quantum field theories can be constructed on the noncommutative spacetime $\mathcal{A}_\theta$ by replacing ordinary multiplication between the fields by $\star$-multiplication and deforming statistics as we discuss below [22 − 25]. These theories are invariant under the deformed Poincaré action [17, 21, 25, 24] under which $\theta_{\mu\nu}$ is invariant. It is thus possible to observe $\theta_{\mu\nu}$ without violating deformed Poincaré symmetry. But of course they are not invariant under the standard undeformed action of the Poincaré group as shown for example by the observability of $\theta_{\mu\nu}$.

3. Quantum fields in noncommutative spacetime

It can be shown immediately that the action of the deformed coproduct is not compatible with standard statistics [25]. Thus for $\theta^{\mu\nu} = 0$, we have the axiom in quantum theory that the statistics operator τ_0 defined by

$$\tau_0\,(\phi \otimes \chi) = \chi \otimes \phi \tag{3.1}$$

is superselected. In particular, the Lorentz group action must and *does* commute with the statistics operator,

$$\tau_0 \Delta_0(\Lambda) = \Delta_0(\Lambda)\tau_0, \tag{3.2}$$

where $\Lambda \in \mathcal{P}_+^\uparrow$, the connected component of the Poincaré group.

Also all the states in a given superselection sector are eigenstates of τ_0 with the same eigenvalue. Given an element $\phi \otimes \chi$ of the tensor product, the physical Hilbert spaces can be constructed from the elements

$$\left(\frac{1 \pm \tau_0}{2}\right)(\phi \otimes \chi). \tag{3.3}$$

Now since $\tau_0 \mathcal{F}_\theta = \mathcal{F}_\theta^{-1}\tau_0$, we have that

$$\tau_0 \Delta_\theta(\Lambda) \neq \Delta_\theta(\Lambda)\tau_0 \tag{3.4}$$

showing that the use of the usual statistics operator is not compatible with the deformed coproduct.

But the new statistics operator

$$\tau_\theta \equiv \mathcal{F}_\theta^{-1}\tau_0\mathcal{F}_\theta, \quad \tau_\theta^2 = 1 \otimes 1 \tag{3.5}$$

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does commute with the deformed coproduct.

The two-particle state $|p,q\rangle_{S_\theta,A_\theta}$ for bosons and fermions obeying deformed statistics is constructed as follows:

$$
\begin{aligned}
|p,q\rangle_{S_\theta,A_\theta} &= |p\rangle \otimes_{S_\theta,A_\theta} |q\rangle = \left(\frac{1\pm\tau_\theta}{2}\right)(|p\rangle\otimes|q\rangle) \\
&= \frac{1}{2}\Big(|p\rangle\otimes|q\rangle \pm \mathrm{e}^{-ip_\mu\theta^{\mu\nu}q_\nu}|q\rangle\otimes|p\rangle\Big).
\end{aligned} \tag{3.6}
$$

Exchanging p and q in the above, one finds

$$
|p,q\rangle_{S_\theta,A_\theta} = \pm\, \mathrm{e}^{-ip_\mu\theta^{\mu\nu}q_\nu}|q,p\rangle_{S_\theta,A_\theta}. \tag{3.7}
$$

In Fock space, the above two-particle state is constructed from a second-quantized field φ_θ according to

$$
\begin{aligned}
\frac{1}{2}\langle 0|\varphi_\theta(x_1)\varphi_\theta(x_2)a^\dagger_{\mathbf{q}}a^\dagger_{\mathbf{p}}|0\rangle &= \left(\frac{1\pm\tau_\theta}{2}\right)(e_p\otimes e_q)(x_1,x_2) \\
&= (e_p\otimes_{S_\theta,A_\theta} e_q)(x_1,x_2) \\
&= \langle x_1,x_2|p,q\rangle_{S_\theta,A_\theta}
\end{aligned} \tag{3.8}
$$

where φ_0 is a boson(fermion) field associated with $|p,q\rangle_{S_0}$ ($|p,q\rangle_{A_0}$).

On using eq. (3.7), this leads to the commutation relation

$$
a^\dagger_{\mathbf{p}}a^\dagger_{\mathbf{q}} = \pm\, \mathrm{e}^{ip_\mu\theta^{\mu\nu}q_\nu}\; a^\dagger_{\mathbf{q}}a^\dagger_{\mathbf{p}}. \tag{3.9}
$$

Let P_μ be the Fock space momentum operator. (It is the representation of the translation generator introduced previously. We use the same symbol for both.) Then the operators $a_{\mathbf{p}}$, $a^\dagger_{\mathbf{p}}$ can be written as follows:

$$
a_{\mathbf{p}} = c_{\mathbf{p}}\,\mathrm{e}^{-\frac{i}{2}p_\mu\theta^{\mu\nu}P_\nu}, \quad a^\dagger_{\mathbf{p}} = c^\dagger_{\mathbf{p}}\,\mathrm{e}^{\frac{i}{2}p_\mu\theta^{\mu\nu}P_\nu}\;, \tag{3.10}
$$

$c_{\mathbf{p}}$'s being $\theta^{\mu\nu}=0$ annihilation operators.

The map from $c_{\mathbf{p}}, c^\dagger_{\mathbf{p}}$ to $a_{\mathbf{p}}, a^\dagger_{\mathbf{p}}$ in eq. (3.10) is known as the "dressing transformation" [26, 27].

In the noncommutative case, a free spin-zero quantum scalar field of mass m has the mode expansion

$$
\varphi_\theta(x) = \int \frac{d^3p}{(2\pi)^3}\,\Big(a_{\mathbf{p}}\,\mathrm{e}_p(x) + a^\dagger_{\mathbf{p}}\,\mathrm{e}_{-p}(x)\Big) \tag{3.11}
$$

where

$$
\mathrm{e}_p(x) = \mathrm{e}^{-i\,p\cdot x}, \quad p\cdot x = p_0x_0 - \mathbf{p}\cdot\mathbf{x}, \quad p_0 = \sqrt{\mathbf{p}^2+m^2} > 0.
$$

The deformed quantum field φ_θ differs form the undeformed quantum field φ_0 in two ways: *i.*) e_p belongs to the noncommutative algebra of $\mathcal{M}^4$ and *ii.*) $a_{\mathbf{p}}$ is deformed by statistics. The deformed statistics can be accounted for by writing [28]

$$
\varphi_\theta = \varphi_0\,\mathrm{e}^{\frac{1}{2}\overleftarrow{\partial}\wedge P} \tag{3.12}
$$

where

$$\overleftarrow{\partial} \wedge P \equiv \overleftarrow{\partial}_\mu \theta^{\mu\nu} P_\nu. \tag{3.13}$$

It is easy to write down the n-point correlation function for the deformed quantum field $\varphi_\theta(x)$ in terms of the undeformed field $\varphi_0(x)$:

$$\langle 0|\varphi_\theta(x_1)\varphi_\theta(x_2)\cdots\varphi_\theta(x_n)|0\rangle = \langle 0|\varphi_0(x_1)\varphi_0(x_2)\cdots\varphi_0(x_n)|0\rangle \; \mathrm{e}^{(-\frac{i}{2}\sum_{J=2}^{n}\sum_{I=1}^{J-1}\overleftarrow{\partial}_{x_I}\wedge\overleftarrow{\partial}_{x_J})}. \tag{3.14}$$

On using

$$\varphi_\theta(x) = \varphi_\theta(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3}\, \Phi_\theta(\mathbf{k}, t)\; \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}, \tag{3.15}$$

we find for the vacuum expectation values, in momentum space

$$\begin{aligned}\langle 0|\Phi_\theta(\mathbf{k}_1, t_1)\Phi_\theta(\mathbf{k}_2, t_2)\cdots\Phi_\theta(\mathbf{k}_n, t_n)|0\rangle = \mathrm{e}^{(\frac{i}{2}\sum_{J>I}\mathbf{k}_I\wedge\mathbf{k}_J)}\langle 0| \\ \times\Phi_0\left(\mathbf{k}_1, t_1 + \frac{\vec{\theta^0}\cdot\mathbf{k}_2 + \vec{\theta^0}\cdot\mathbf{k}_3 + \cdots + \vec{\theta^0}\cdot\mathbf{k}_n}{2}\right) \\ \times\Phi_0\left(\mathbf{k}_2, t_2 + \frac{-\vec{\theta^0}\cdot\mathbf{k}_1 + \vec{\theta^0}\cdot\mathbf{k}_3 + \cdots + \vec{\theta^0}\cdot\mathbf{k}_n}{2}\right)\cdots \\ \times\Phi_0\left(\mathbf{k}_n, t_n + \frac{-\vec{\theta^0}\cdot\mathbf{k}_1 - \vec{\theta^0}\cdot\mathbf{k}_2 - \cdots - \vec{\theta^0}\cdot\mathbf{k}_{n-1}}{2}\right)|0\rangle\end{aligned} \tag{3.16}$$

where

$$\vec{\theta^0} = (\theta^{01}, \theta^{02}, \theta^{03}). \tag{3.17}$$

Since the underlying Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime has spatial translational invariance,

$$\mathbf{k}_1 + \mathbf{k}_2 + \cdots + \mathbf{k}_n = 0,$$

the n-point correlation function in momentum space becomes

$$\begin{aligned}\langle 0|\Phi_\theta(\mathbf{k}_1, t_1)\Phi_\theta(\mathbf{k}_2, t_2)\cdots\Phi_\theta(\mathbf{k}_n, t_n)|0\rangle = \mathrm{e}^{(\frac{i}{2}\sum_{J>I}\mathbf{k}_I\wedge\mathbf{k}_J)}\langle 0|\Phi_0\left(\mathbf{k}_1, t_1 - \frac{\vec{\theta^0}\cdot\mathbf{k}_1}{2}\right) \\ \times\Phi_0\left(\mathbf{k}_2, t_2 - \vec{\theta^0}\cdot\mathbf{k}_1 - \frac{\vec{\theta^0}\cdot\mathbf{k}_2}{2}\right) \\ \times\cdots\Phi_0\left(\mathbf{k}_n, t_n - \vec{\theta^0}\cdot\mathbf{k}_1 - \vec{\theta^0}\cdot\mathbf{k}_2 - \cdots - \vec{\theta^0}\cdot\mathbf{k}_{n-1} - \frac{\vec{\theta^0}\cdot\mathbf{k}_n}{2}\right)|0\rangle.\end{aligned} \tag{3.18}$$

In particular, the two-point correlation function is

$$\langle 0|\Phi_\theta(\mathbf{k}_1, t_1)\Phi_\theta(\mathbf{k}_2, t_2)|0\rangle = \langle 0|\Phi_0\left(\mathbf{k}_1, t_1 - \frac{\vec{\theta^0}\cdot\mathbf{k}_1}{2}\right)\Phi_0\left(\mathbf{k}_2, t_2 - \frac{\vec{\theta^0}\cdot\mathbf{k}_1}{2}\right)|0\rangle, \tag{3.19}$$

since it vanishes unless $\mathbf{k}_1 + \mathbf{k}_2 = 0$ and hence $\mathrm{e}^{(\frac{i}{2}\sum_{J>I}\; k_I\wedge k_J)} = 1$.

We emphasize that eqs. (3.16), (3.18) and (3.19) come from eq. (3.9) which implies eqs. (3.10), (3.12) and (3.14). They are exclusively due to deformed statistics. The $*$-product is still mandatory when taking products of φ_θ evaluated at the same point.

In standard Hopf algebra theory, the exchange operation is to be performed using the $\mathcal{R}$-matrix times the flip operator σ [29, 30]. It is easy to check that $\mathcal{R}\sigma$ acts as identity on any pair of factors in eqs. (3.16) and (3.18).

One can also explicitly show that the n-point functions are invariant under the twisted Poincaré group while those of the conventional theory are not. Hence the requirement of twisted Poincaré invariance fixes the structure of n-point functions. These points are discussed further in [25].

It is interesting to note that the two-point correlation function is nonlocal in time in the noncommutative frame work. Also note the following: Assuming that $\theta^{\mu\nu}$ is non-degenerate, we can write it as

$$\begin{aligned}&\theta^{\mu\nu} = \alpha\, \epsilon_{ab}\, e_a^\mu\, e_b^\nu + \beta\, \epsilon_{ab}\, f_a^\mu\, f_b^\nu, \\ &\alpha, \beta \neq 0, \qquad\qquad \epsilon_{ab} = -\epsilon_{ba}, \qquad\qquad a, b = 1, 2\end{aligned}$$

where e_a, e_b, f_a, f_b are orthonormal real vectors. Thus $\theta^{\mu\nu}$ defines two distinguished two-planes in $\mathcal{M}^4$, namely those spanned by e_a and by f_a. For simplicity we have assumed that one of these planes contains the time direction, say $e_1 : e_1^\mu = \delta_0^\mu$. The θ^{0i} part then can be regarded as defining a spatial direction $\vec{\theta}^0$ as given by eq. (3.17).

We will make use of the modified two-point correlation functions given by eq. (3.19) when we define the power spectrum for inflaton field perturbations in the noncommutative frame work.

4. Cosmological perturbations and (direction-independent) power spectrum for $\theta^{\mu\nu} = 0$

In this section we briefly review how fluctuations in the inflaton field cause inhomogeneities in the distribution of matter and radiation following [31].

The scalar field ϕ driving inflation can be split into a zeroth order homogeneous part and a first order perturbation:

$$\phi(\mathbf{x}, t) = \phi^{(0)}(t) + \delta\phi(\mathbf{x}, t) \tag{4.1}$$

The energy-momentum tensor for ϕ is

$$\mathcal{T}^\alpha{}_\beta = g^{\alpha\nu} \frac{\partial\phi}{\partial x^\nu} \frac{\partial\phi}{\partial x^\beta} - g^\alpha{}_\beta \left[\frac{1}{2} g^{\mu\nu} \frac{\partial\phi}{\partial x^\mu} \frac{\partial\phi}{\partial x^\nu} + V(\phi)\right] \tag{4.2}$$

We assume a spatially flat, homogeneous and isotropic (FLRW) background with the metric

$$ds^2 = dt^2 - \mathfrak{a}^2(t) d\mathbf{x}^2 \tag{4.3}$$

where $\mathfrak{a}$ is the cosmological scale factor, and nonvanishing Γ's

$$\Gamma^0{}_{ij} = \delta_{ij} \mathfrak{a}^2 H \quad \text{and} \quad \Gamma^i{}_{0j} = \Gamma^i{}_{j0} = \delta^i_j H$$

where H is the Hubble parameter.

In conformal time η where $d\eta = \frac{dt}{\mathfrak{a}(t)}$, $-\infty < \eta < 0$, the metric becomes

$$ds^2 = a^2(\eta)\left(d\eta^2 - d\mathbf{x}^2\right), \tag{4.4}$$

where a is the cosmological scale factor now regarded as a function of conformal time. Using this metric we write the equation for the zeroth order part of ϕ [31],

$$\ddot{\phi}^{(0)} + 2aH\dot{\phi}^{(0)} + a^2 V'\phi^{(0)} = 0, \tag{4.5}$$

where overdots denote derivatives with respect to conformal time η and V' is the derivative of V with respect to the field $\phi^{(0)}$. Notice that in conformal time η we have $\frac{da(\eta)}{d\eta} = a^2(\eta)H$ while in cosmic time t we have $\frac{d\mathfrak{a}(t)}{dt} = \mathfrak{a}H$.

The equation for $\delta\phi$ can be obtained from the first order perturbation of the energy-momentum tensor conservation equation:

$$\mathcal{T}^{\mu}{}_{\nu;\,\mu} = \frac{\partial \mathcal{T}^{\mu}{}_{\nu}}{\partial x^{\mu}} + \Gamma^{\mu}{}_{\alpha\mu}\mathcal{T}^{\alpha}{}_{\nu} - \Gamma^{\alpha}{}_{\nu\mu}\mathcal{T}^{\mu}{}_{\alpha} = 0. \tag{4.6}$$

The perturbed part of the energy-momentum tensor $\delta T^{\mu}{}_{\nu}$ satisfies the following conservation equation in momentum space [31]:

$$\frac{\partial \delta T^{0}{}_{0}}{\partial t} + ik_i \delta T^{i}{}_{0} + 3H\delta T^{0}{}_{0} - H\delta T^{i}{}_{i} = 0, \tag{4.7}$$

where

$$T^{\mu\nu}(\mathbf{k},t) = \int d^3x\ \mathcal{T}^{\mu\nu}(\mathbf{x},t)\ \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{4.8}$$

Let $\phi(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^3}\ \tilde{\phi}(\mathbf{k},t)\ \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}$. Writing down the perturbations to the energy-momentum tensor in terms of $\tilde{\phi}(\mathbf{k},t)$,

$$\begin{aligned}
\delta T^{i}{}_{0} &= \frac{ik_i}{a^3}\dot{\tilde{\phi}}^{(0)}\delta\tilde{\phi}, \\
\delta T^{0}{}_{0} &= \frac{-\dot{\tilde{\phi}}^{(0)}\delta\dot{\tilde{\phi}}}{a^2} - V'\left(\tilde{\phi}^{(0)}\right)\delta\tilde{\phi}, \\
\delta T^{i}{}_{j} &= \delta_{ij}\left(\frac{\dot{\tilde{\phi}}^{(0)}\delta\dot{\tilde{\phi}}}{a^2} - V'\left(\tilde{\phi}^{(0)}\right)\delta\tilde{\phi}\right),
\end{aligned}$$

the conservation equation becomes

$$\delta\ddot{\tilde{\phi}} + 2aH\delta\dot{\tilde{\phi}} + k^2\delta\tilde{\phi} = 0. \tag{4.9}$$

Eliminating the middle Hubble damping term by a change of variable $\zeta(\mathbf{k},\eta) = a(\eta)\delta\tilde{\phi}(\mathbf{k},\eta)$, the above equation becomes

$$\ddot{\zeta}(\mathbf{k},\eta) + \omega_k^2(\eta)\zeta(\mathbf{k},\eta) = 0, \quad \omega_k^2(\eta) \equiv \left(k^2 - \frac{\ddot{a}(\eta)}{a(\eta)}\right). \tag{4.10}$$

The mode functions u associated with the quantum operator $\hat{\zeta}$ satisfy

$$\ddot{u}(\mathbf{k},\eta) + \left(k^2 - \frac{\ddot{a}(\eta)}{a(\eta)}\right) u(\mathbf{k},\eta) = 0 \tag{4.11}$$

with the initial conditions $u(\mathbf{k},\eta_i) = \frac{1}{\sqrt{2\omega_k(\eta_i)}}$ and $\dot{u}(\mathbf{k},\eta_i) = i\sqrt{\omega_k(\eta_i)}$. Notice that these initial conditions have meaning only when $\omega_k(\eta_i) > 0$.

We can immediately write down the quantum operator associated with the variable ζ,

$$\hat{\zeta}(\mathbf{k},\eta) = u(\mathbf{k},\eta)\hat{a}_{\mathbf{k}} + u^*(\mathbf{k},\eta)\hat{a}^\dagger_{\mathbf{k}}, \tag{4.12}$$

with the bosonic commutation relations $[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = 0$ and $[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')$.

During inflation we have scale factor $a(\eta) \simeq -(\eta H)^{-1}$. Thus eq. (4.11) takes the form [31]

$$\ddot{u} + \left(k^2 - \frac{2}{\eta^2}\right) u = 0. \tag{4.13}$$

When the perturbation modes are well within the horizon, $k|\eta| \gg 1$, one can obtain a properly normalized solution $u(\mathbf{k},\eta)$ from the conditions imposed on it at very early times during inflation. Such a solution is [31, 32]

$$u(\mathbf{k},\eta) = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right) \mathrm{e}^{-ik(\eta-\eta_i)}. \tag{4.14}$$

The variances involving $\hat{\zeta}$ and $\hat{\zeta}^\dagger$ are

$$\begin{aligned}
\langle 0|\hat{\zeta}(\mathbf{k},\eta)\hat{\zeta}(\mathbf{k}',\eta)|0\rangle &= 0, \\
\langle 0|\hat{\zeta}^\dagger(\mathbf{k},\eta)\hat{\zeta}^\dagger(\mathbf{k}',\eta)|0\rangle &= 0, \\
\langle 0|\hat{\zeta}^\dagger(\mathbf{k},\eta)\hat{\zeta}(\mathbf{k}',\eta)|0\rangle &= (2\pi)^3|u(\mathbf{k},\eta)|^2\delta^3(\mathbf{k}-\mathbf{k}') \\
&\equiv (2\pi)^3 P_\zeta(\mathbf{k},\eta)\delta^3(\mathbf{k}-\mathbf{k}')
\end{aligned} \tag{4.15}$$

where P_ζ is the power spectrum of $\hat{\zeta}$. Eq. (4.15) can be treated as a general definition of power spectrum.

In the case when spacetime is commutative ($\theta^{\mu\nu} = 0$), the power spectrum in eq. (4.15) is

$$\langle 0|\hat{\zeta}^\dagger(\mathbf{k},\eta)\hat{\zeta}(\mathbf{k}',\eta)|0\rangle = (2\pi)^3 P_\zeta(k,\eta)\delta^3(\mathbf{k}-\mathbf{k}'). \tag{4.16}$$

The Dirac delta function in eqs. (4.15) and (4.16) shows that perturbations with different wave numbers are uncoupled as a consequence of the translational invariance of the underlying spacetime. Rotational invariance of the underlying (commutative) spacetime constraints the power spectrum $P_\zeta(k,\eta)$ to depend only on the magnitude of $\mathbf{k}$.

Towards the end of inflation, $k|\eta|$ ($-\infty < \eta < 0$) becomes very small. In that case the small argument limit of eq. (4.14),

$$\lim_{k|\eta|\to 0} u(\mathbf{k},\eta) = \frac{1}{\sqrt{2k}}\,\frac{-i}{k\eta}\,\mathrm{e}^{-ik(\eta-\eta_i)}, \tag{4.17}$$

gives the power spectrum $P_\zeta(k,\eta) = |u(\mathbf{k},\eta)|^2$. On using $\zeta(\mathbf{k},\eta) = a(\eta)\delta\tilde{\phi}(\mathbf{k},\eta)$, we write the power spectrum $P_{\delta\tilde{\phi}}$ for the scalar field perturbations [31]:

$$P_{\delta\tilde{\phi}}(k,\eta) = \frac{|u(\mathbf{k},\eta)|^2}{a(\eta)^2} = \frac{1}{2k^3}\frac{1}{a(\eta)^2\eta^2}. \tag{4.18}$$

In terms of the Hubble parameter H during inflation ($H \simeq -\frac{1}{a(\eta)\eta}$), the power spectrum becomes

$$P_{\delta\tilde{\phi}}(k,\eta) = \frac{1}{2k^3}H^2. \tag{4.19}$$

We are interested in the post-inflation power spectrum for the scalar metric perturbations since they couple to matter and radiation and give rise to inhomogeneities and anisotropies in their respective distributions which we observe. This spectrum comes from the inflaton field since the inflaton field perturbations get transferred to the scalar part of the metric.

We write the perturbed metric in the longitudinal gauge [33],

$$ds^2 = a^2(\eta)\Big[(1+2\chi(\mathbf{x},\eta))d\eta^2 - (1-2\Psi(\mathbf{x},\eta))\gamma^{ij}(\mathbf{x},\eta)dx_i dx_j\Big], \tag{4.20}$$

where χ and Ψ are two physical metric degrees of freedom describing the scalar metric perturbations and γ^{ij} is the metric of the unperturbed spatial hypersurfaces.

In our model, as in the case of most simple cosmological models, in the absence of anisotropic stress ($\delta T^i{}_j = 0$ for $i \neq j$), the two scalar metric degrees of freedom χ and Ψ coincide upto a sign:

$$\Psi = -\chi. \tag{4.21}$$

The remaining metric perturbation Ψ can be expressed in terms of the inflaton field fluctuation $\delta\tilde{\phi}$ at horizon crossing [31],

$$\tilde{\Psi}\Big|_{\text{post inflation}} = \frac{2}{3}aH\frac{\delta\tilde{\phi}}{\dot{\tilde{\phi}}^{(0)}}\Big|_{\text{horizon crossing}} \tag{4.22}$$

where $\tilde{\Psi}$ is the Fourier coefficient of Ψ.

On using the general definition of power spectrum as in eq. (4.16), the power spectra for $P_{\tilde{\Psi}}$ and $P_{\delta\tilde{\phi}}$ can be connected when a mode k crosses the horizon, i.e. when $a(\eta)H = k$, say for $\eta = \eta_0$:

$$P_{\tilde{\Psi}}(\mathbf{k},\eta) = \frac{4}{9}\left(\frac{a(\eta)H}{\dot{\tilde{\phi}}^{(0)}}\right)^2 P_{\delta\tilde{\phi}}\Big|_{a(\eta_0)H=k}. \tag{4.23}$$

From eq. (4.19), eq. (4.21) and using

$$aH/\dot{\tilde{\phi}}^{(0)} = \sqrt{4\pi G/\epsilon} \tag{4.24}$$

at horizon crossing, where G is Newton's gravitational constant and ϵ is the slow-roll parameter in the single field inflation model [31], we have the power spectrum (defined as in eq. (4.16)) for the scalar metric perturbation at horizon crossing,

$$P_{\tilde{\Psi}}(k,\eta(t)) = P_{\Phi_0}(k,\eta(t)) = \frac{16\pi G}{9\epsilon}\frac{H^2}{2k^3}\Big|_{a(\eta_0)H=k}, \tag{4.25}$$

Here we wrote Φ_0 for $\tilde{\chi}$.

Note that the Hubble parameter H is (nearly) constant during inflation and also it is the same in conformal time η and cosmic time t. Since the time dependence of the power spectrum is through the Hubble parameter in eq. (4.25), we have

$$P_{\Phi_0}(k, \eta(t)) = P_{\Phi_0}(k, t) \equiv P_{\Phi_0}(k) = \text{constant in time.} \tag{4.26}$$

The power spectrum in eq. (4.25) is for commutative spacetime and it depends on the magnitude of $\mathbf{k}$ and not on its direction. In the next section, we will show that the power spectrum becomes direction-dependent when we make spacetime noncommutative.

5. Direction-dependent power spectrum

The two-point function in noncommutative spacetime, using eq. (3.19), takes the form

$$\langle 0|\Phi_\theta(\mathbf{k}, \eta)\Phi_\theta(\mathbf{k}', \eta)|0\rangle = \langle 0|\Phi_0(\mathbf{k}, \eta^-)\Phi_0(\mathbf{k}', \eta^-)|0\rangle\,, \tag{5.1}$$

where $\eta^- = \eta\left(t - \frac{\vec{\theta}^0\cdot\mathbf{k}}{2}\right)$.

In the commutative case, the reality of the two-point correlation function (since the density fields Φ_0 are real) is obtained by imposing the condition

$$\langle \Phi_0(\mathbf{k}, \eta)\Phi_0(\mathbf{k}', \eta)\rangle^* = \langle \Phi_0(-\mathbf{k}, \eta)\Phi_0(-\mathbf{k}', \eta)\rangle. \tag{5.2}$$

But this condition is not correct when the fields are deformed. That is because even if Φ_θ is self-adjoint, $\Phi_\theta(\mathbf{x}, t)\Phi_\theta(\mathbf{x}', t') \neq \Phi_\theta(\mathbf{x}', t')\Phi_\theta(\mathbf{x}, t)$ for space-like separations. A simple and natural modification (denoted by subscript M) of the correlation function that ensures reality involves "symmetrization" of the product of φ_θ's or keeping its self-adjoint part. That involves replacing the product of ϕ_θ's by half its anti-commutator,

$$\frac{1}{2}[\varphi_\theta(\mathbf{x}, \eta), \varphi_\theta(\mathbf{y}, \eta)]_+ = \frac{1}{2}\left(\varphi_\theta(\mathbf{x}, \eta)\varphi_\theta(\mathbf{y}, \eta) + \varphi_\theta(\mathbf{y}, \eta)\varphi_\theta(\mathbf{x}, \eta)\right). \tag{5.3}$$

(We emphasize that this procedure for ensuring reality is a matter of choice)

For the Fourier modes Φ_θ, this procedure gives:

$$\langle \Phi_\theta(\mathbf{k}, \eta)\Phi_\theta(\mathbf{k}', \eta)\rangle_M = \frac{1}{2}\left(\langle \Phi_\theta(\mathbf{k}, \eta)\Phi_\theta(\mathbf{k}', \eta)\rangle + \langle \Phi_\theta(-\mathbf{k}, \eta)\Phi_\theta(-\mathbf{k}', \eta)\rangle^*\right) \tag{5.4}$$

After the modification of the correlation function, the power spectrum for scalar metric perturbation takes the form

$$\langle \Phi_\theta(\mathbf{k}, \eta)\Phi_\theta(\mathbf{k}', \eta)\rangle_M = (2\pi)^3 P_{\Phi_\theta}(\mathbf{k}, \eta)\delta^3(\mathbf{k} + \mathbf{k}'). \tag{5.5}$$

Using eqs. (4.18), (4.23), (5.1) and (5.4) we write down the modified power spectrum:

$$P_{\Phi_\theta}(\mathbf{k}, \eta) = \frac{1}{2}\left[\frac{4}{9}\left(\frac{a(\eta)H}{\dot{\tilde{\phi}}^{(0)}}\right)^2 \frac{1}{a(\eta)^2}\left(|u(\mathbf{k}, \eta^-)|^2 + |u(-\mathbf{k}, \eta^+)|^2\right)\right]. \tag{5.6}$$

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where $\eta^{\pm} = \eta(t \pm \frac{\vec{\theta}^0 \cdot \mathbf{k}}{2})$. Notice that here the argument of the scale factor $a(\eta)$ is not shifted, since it is not deformed by noncommutativity.

It is easy to show that

$$u(\mathbf{k}, \eta^{\pm}) = \frac{e^{-ik\eta^{\pm}}}{\sqrt{2k}} \left(1 - \frac{i}{k\eta^{\pm}}\right) \tag{5.7}$$

are also solutions of eq. (4.13).

Thus on using eq. (4.24) and the limit $k\eta^{\pm} \to 0$ of eq. (5.7), the modified power spectrum is found to be

$$\begin{aligned} P_{\Phi_\theta}(\mathbf{k}, \eta) &= \frac{1}{2} \left[\frac{16\pi G}{9\epsilon} \frac{1}{a(\eta)^2} \left(|u(\mathbf{k}, \eta^-)|^2 + |u(-\mathbf{k}, \eta^+)|^2 \right) \right] \\ &= \frac{1}{2} \left[\frac{16\pi G}{9\epsilon} \frac{1}{a(\eta)^2} \left(\frac{1}{2k^3(\eta^-)^2} + \frac{1}{2k^3(\eta^+)^2} \right) \right] \\ &= \frac{8\pi G}{9\epsilon} \frac{1}{2k^3 a(\eta)^2} \left(\frac{1}{(\eta^-)^2} + \frac{1}{(\eta^+)^2} \right). \end{aligned} \tag{5.8}$$

Assuming that the Hubble parameter H is nearly a constant during inflation, the conformal time [31]

$$\eta(t) \simeq \frac{-1}{Ha_0}\, e^{-Ht}. \tag{5.9}$$

gives an expression for $\eta^{\pm}$:

$$\eta^{\pm} = \eta(t)\, e^{\mp \frac{1}{2} H \vec{\theta}^0 \cdot \mathbf{k}}. \tag{5.10}$$

On using eq. (5.10) in eq. (5.8) we can easily write down an analytic expression for the modified primordial power spectrum at horizon crossing,

$$P_{\Phi_\theta}(\mathbf{k}) = P_{\Phi_0}(k) \cosh\left(H\vec{\theta}^0 \cdot \mathbf{k}\right) \tag{5.11}$$

where $P_{\Phi_0}(k)$ is given by eq. (4.25). Note that the modified power spectrum also respects the $\mathbf{k} \to -\mathbf{k}$ parity symmetry.

This power spectrum depends on both the magnitude and direction of $\mathbf{k}$ and clearly breaks rotational invariance. In the next section we will connect this power spectrum to the two-point temperature correlations in the sky and obtain an expression for the amount of deviation from statistical isotropy due to noncommutativity.

6. Signature of noncommutativity in the CMB radiation

We are interested in quantifying the effects of noncommutative scalar perturbations on the cosmic microwave background fluctuations. We assume homogeneity of temperature fluctuations observed in the sky. Hence it is a function of a unit vector giving the direction in the sky and can be expanded in spherical harmonics:

$$\frac{\Delta T(\hat{n})}{T} = \sum_{lm} a_{lm} Y_{lm}(\hat{n}), \tag{6.1}$$

Here $\hat{n}$ is the direction of incoming photons.

The coefficients of spherical harmonics contain all the information encoded in the temperature fluctuations. For $\theta^{\mu\nu} = 0$, they can be connected to the primordial scalar metric perturbations Φ_0,

$$a_{lm} = 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3}\, \Delta_l(k)\Phi_0(\mathbf{k},\eta)Y^*_{lm}(\hat{k}), \tag{6.2}$$

where $\Delta_l(k)$ are called *transfer functions*. They describe the evolutions of scalar metric perturbations Φ_0 from horizon crossing epoch to a time well into the radiation dominated epoch.

The two-point temperature correlation function can be expanded in spherical harmonics:

$$\left\langle \frac{\Delta T(\hat{n})}{T}\frac{\Delta T(\hat{n}')}{T}\right\rangle = \sum_{lml'm'} \langle a_{lm}a^*_{l'm'}\rangle Y^*_{lm}(\hat{n})Y_{l'm'}(\hat{n}'). \tag{6.3}$$

The variance of a_{lm}'s is nonzero. For $\theta^{\mu\nu} = 0$, we have

$$\langle a_{lm}a^*_{l'm'}\rangle = C_l\delta_{ll'}\delta_{mm'}. \tag{6.4}$$

Using eq. (4.16) and eq. (6.2), we can derive the expression for C_l's for $\theta^{\mu\nu} = 0$:

$$\begin{aligned}\langle a_{lm}a^*_{l'm'}\rangle &= 16\pi^2(-i)^{l-l'}\int\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\,\Delta_l(k)\Delta_{l'}(k')\langle\Phi_0(\mathbf{k},\eta)\Phi_0^*(\mathbf{k}',\eta)\rangle\; Y^*_{lm}(\hat{k})Y_{l'm'}(\hat{k}')\\ &= 16\pi^2(-i)^{l-l'}\int\frac{d^3k}{(2\pi)^3}\,\Delta_l(k)\Delta_{l'}(k)P_{\Phi_0}(k)\; Y^*_{lm}(\hat{k})Y_{l'm'}(\hat{k})\\ &= \frac{2}{\pi}\int dk\; k^2\;(\Delta_l(k))^2\; P_{\Phi_0}(k)\;\delta_{ll'}\delta_{mm'} = C_l\;\delta_{ll'}\delta_{mm'},\end{aligned} \tag{6.5}$$

where $P_{\Phi_0}(k)$ is given by eq. (4.25).

When the fields are noncommutative, the two-point temperature correlation function clearly depends on $\theta^{\mu\nu}$. We can still write the two-point temperature correlation as in eq. (6.3):

$$\left\langle \frac{\Delta T(\hat{n})}{T}\frac{\Delta T(\hat{n}')}{T}\right\rangle_\theta = \sum_{lml'm'} \langle a_{lm}a^*_{l'm'}\rangle_\theta Y_{lm}(\hat{n})Y^*_{l'm'}(\hat{n}'). \tag{6.6}$$

This gives

$$\langle a_{lm}a^*_{l'm'}\rangle_\theta = 16\pi^2(-i)^{l-l'}\int\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\Delta_l(k)\Delta_{l'}(k')\langle\Phi_\theta(\mathbf{k},\eta)\Phi_\theta^\dagger(\mathbf{k}',\eta)\rangle_M Y^*_{lm}(\hat{k})Y_{l'm'}(\hat{k}'). \tag{6.7}$$

The two-point correlation function in eq. (6.7) is calculated during the horizon crossing of the mode $\mathbf{k}$. Once a mode crosses the horizon, it becomes independent of time, so that we can rewrite the two-point function as

$$\langle\Phi_\theta(\mathbf{k},\eta)\Phi_\theta^\dagger(\mathbf{k}',\eta)\rangle_M = (2\pi)^3P_{\Phi_\theta}(\mathbf{k})\delta^3(\mathbf{k}-\mathbf{k}') \tag{6.8}$$

where $P_{\Phi_\theta}(\mathbf{k})$ is given by eq. (5.11).

Thus we write the noncommutative angular correlation function as follows:

$$\langle a_{lm}a^*_{l'm'}\rangle_\theta \; = 16\pi^2(-i)^{l-l'}\int \frac{d^3k}{(2\pi)^3}\,\Delta_l(k)\Delta_{l'}(k)P_{\Phi_\theta}(\mathbf{k})\; Y^*_{lm}(\hat{k})Y_{l'm'}(\hat{k}). \tag{6.9}$$

The regime in which the transfer functions act is well above the noncommutative length scale, so that it is perfectly legitimate to assume that the transfer functions are the same as in the commutative case.

Assuming that the $\vec{\theta^0}$ is along the z-axis, we have the expansion

$$\mathrm{e}^{\pm H\vec{\theta^0}\cdot\mathbf{k}} = \sum_{l=0}^{\infty} i^l\sqrt{4\pi(2l+1)}j_l(\mp i\theta kH)Y_{l0}(\cos\vartheta) \tag{6.10}$$

where $\vec{\theta^0}\cdot\mathbf{k} = \theta k\cos\vartheta$ and j_l is the spherical Bessel function.

On using eq. (6.10) and the identities $j_l(-z) = (-1)^l j_l(z)$ and $j_l(iz) = i^l\; i_l(z)$, where i_l is the modified spherical Bessel function, we can write eq. (5.11) as

$$P_{\Phi_\theta}(\mathbf{k}) = P_{\Phi_0}(k)\sum_{l=0,\; l:\mathrm{even}}^{\infty}\sqrt{4\pi(2l+1)}\; i_l(\theta kH)\; Y_{l0}(\cos\vartheta). \tag{6.11}$$

Using eqs. (6.9) and (6.11), we rewrite eq. (6.9) as,

$$\begin{aligned}\langle a_{lm}a^*_{l'm'}\rangle_\theta \; &= \frac{2}{\pi}\int dk \sum_{l''=0,\; l'':\mathrm{even}}^{\infty} (i)^{l-l'}(-1)^m(2l''+1)\; k^2\Delta_l(k)\Delta_{l'}(k)P_{\Phi_0}(k)i_{l''}(\theta kH)\\ &\quad\times\sqrt{(2l+1)(2l'+1)}\begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & l' & l'' \\ -m & m' & 0\end{pmatrix},\end{aligned} \tag{6.12}$$

the Wigner's 3-j symbols in eq. (6.12) being related to the integrals of spherical harmonics:

$$\int d\Omega_k\; Y_{l,-m}(\hat{k})Y_{l'm'}(\hat{k})Y_{l''0}(\hat{k}) = \sqrt{(2l+1)(2l'+1)(2l''+1)/4\pi}\begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & l' & l'' \\ -m & m' & 0\end{pmatrix}. \tag{6.13}$$

We can also get a simplified form of eq. (6.12) by expanding the modified power spectrum in eq. (5.11) in powers of θ up to the leading order:

$$P_{\Phi_\theta}(\mathbf{k}) \simeq P_{\Phi_0}(k)\left[1 + \frac{H^2}{2}\left(\vec{\theta^0}\cdot\mathbf{k}\right)^2\right]. \tag{6.14}$$

A modified power spectrum of this form has been considered in [34], where the rotational invariance is broken by introducing a (small) nonzero vector. In our case, the vector that breaks rotational invariance is $\vec{\theta^0}$ and it emerges naturally in the framework of field theories on the noncommutative Groenewold-Moyal spacetime. We have also an exact expression for $P_{\Phi_\theta}(\mathbf{k})$ in eq. (5.11).

Work is in progress to find a best fit for the data available and thereby to determine the length scale of noncommutativity.

The direction-dependent primordial power spectrum discussed in [34] is considered in a model independent way in [35] to compute minimum-variance estimators for the coefficients

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of direction-dependence. A test for the existence of a preferred direction in the primordial perturbations using full-sky CMB maps is performed in a model independent way in [36]. Imprints of cosmic microwave background anisotropies from a non-standard spinor field driven inflation is considered in [37]. Anisotropic dark energy equation of state can also give rise to a preferred direction in the universe [38].

7. Non-gaussianity from noncommutativity

In this section, we briefly explain how n-point correlation functions become non-Gaussian when the fields are noncommutative, assuming that they are Gaussian in their commutative limits.

Consider a noncommutative field $\varphi_\theta(\mathbf{x}, t)$. Its first moment is obviously zero:

$$\langle \varphi_\theta(\mathbf{x}, t)\rangle = \langle \varphi_0(\mathbf{x}, t)\rangle = 0.$$

The information about noncommutativity is contained in the higher moments of φ_θ. We show that the n-point functions cannot be written as sums of products of two-point functions. That proves that the underlying probability distribution is non-Gaussian.

The n-point correlation function is

$$C_n(x_1, x_2, \ldots, x_n) = \langle \varphi_\theta(\mathbf{x}_1, t_1) \cdots \varphi_\theta(\mathbf{x}_n, t_n)\rangle \tag{7.1}$$

Since φ_0 is assumed to be Gaussian and φ_θ is given in terms of φ_0 by eq. (3.12), all the odd moments of φ_θ vanish.

But the even moments of φ_θ need not vanish and do not split into sums of products of its two-point functions in a familiar way.

Non-Gaussianity cannot be seen at the level of two-point functions. Consider the two-point function C_2. We write this in momentum space in terms of Φ_0:

$$C_2 = \langle \Phi_\theta(\mathbf{k}_1, t_1)\Phi_\theta(\mathbf{k}_2, t_2)\rangle = e^{-\frac{i}{2}(\mathbf{k}_2\wedge\mathbf{k}_1)}\left\langle \Phi_0\left(\mathbf{k}_1, t_1+\frac{\vec{\theta}^0\cdot\mathbf{k}_2}{2}\right)\Phi_0\left(\mathbf{k}_2, t_2-\frac{\vec{\theta}^0\cdot\mathbf{k}_1}{2}\right)\right\rangle. \tag{7.2}$$

where $\mathbf{k}_i \wedge \mathbf{k}_j \equiv k_i\theta^{ij}k_j$.

Making use of the translation invariance $\mathbf{k}_1 + \mathbf{k}_2 = 0$, the above equation becomes

$$\langle \Phi_\theta(\mathbf{k}_1, t_1)\Phi_\theta(\mathbf{k}_2, t_2)\rangle = \left\langle \Phi_0\left(\mathbf{k}_1, t_1 - \frac{\vec{\theta}^0\cdot\mathbf{k}_1}{2}\right)\Phi_0\left(\mathbf{k}_2, t_2 - \vec{\theta}^0\cdot\mathbf{k}_1 - \frac{\vec{\theta}^0\cdot\mathbf{k}_2}{2}\right)\right\rangle. \tag{7.3}$$

Non-Gaussianity can be seen in all the n-point functions for $n \geq 4$ and even n. Still they can all be written in terms of correlation functions of Φ_0. For example, let us consider the four-point function C_4:

$$\begin{aligned} C_4 = \langle \Phi_\theta(\mathbf{k}_1, t_1)\Phi_\theta(\mathbf{k}_2, t_2)\Phi_\theta(\mathbf{k}_3, t_3)\Phi_\theta(\mathbf{k}_4, t_4)\rangle &= e^{-\frac{i}{2}(\mathbf{k}_3\wedge\mathbf{k}_2+\mathbf{k}_3\wedge\mathbf{k}_1+\mathbf{k}_2\wedge\mathbf{k}_1)} \\ \times \left\langle \Phi_0\left(\mathbf{k}_1, t_1-\frac{\vec{\theta}^0\cdot\mathbf{k}_1}{2}\right)\Phi_0\left(\mathbf{k}_2, t_2-\vec{\theta}^0\cdot\mathbf{k}_1-\frac{\vec{\theta}^0\cdot\mathbf{k}_2}{2}\right)\right.&\left.\Phi_0\left(\mathbf{k}_3, t_3-\vec{\theta}^0\cdot\mathbf{k}_1-\vec{\theta}^0\cdot\mathbf{k}_2-\frac{\vec{\theta}^0\cdot\mathbf{k}_3}{2}\right)\right. \\ \left.\times\, \Phi_0\left(\mathbf{k}_4, t_4 - \vec{\theta}^0\cdot\mathbf{k}_1 - \vec{\theta}^0\cdot\mathbf{k}_2 - \vec{\theta}^0\cdot\mathbf{k}_3 - \frac{\vec{\theta}^0\cdot\mathbf{k}_4}{2}\right)\right\rangle & \end{aligned}$$

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Here we have used translational invariance, which implies that $\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4=0$. Using this equation once more to eliminate $\mathbf{k}_4$, we find

$$C_4 = e^{-\frac{i}{2}(\mathbf{k}_3\wedge\mathbf{k}_2+\mathbf{k}_3\wedge\mathbf{k}_1+\mathbf{k}_2\wedge\mathbf{k}_1)} \left\langle \Phi_0\left(\mathbf{k}_1, t_1-\frac{\vec{\theta}^0\cdot\mathbf{k}_1}{2}\right)\Phi_0\left(\mathbf{k}_2, t_2-\vec{\theta}^0\cdot\mathbf{k}_1-\frac{\vec{\theta}^0\cdot\mathbf{k}_2}{2}\right)\times\right.$$
$$\left.\times\Phi_0\left(\mathbf{k}_3, t_3-\vec{\theta}^0\cdot\mathbf{k}_1-\vec{\theta}^0\cdot\mathbf{k}_2-\frac{\vec{\theta}^0\cdot\mathbf{k}_3}{2}\right)\Phi_0\left(\mathbf{k}_4, t_4-\frac{\vec{\theta}^0\cdot\mathbf{k}_1+\vec{\theta}^0\cdot\mathbf{k}_2+\vec{\theta}^0\cdot\mathbf{k}_3}{2}\right)\right\rangle$$

Assuming Gaussianity for the field Φ_0 and denoting $\Phi_0(\mathbf{k}_i, t_i)$ by $\Phi_0^{(i)}$, we have,

$$\left\langle\Phi_0^{(1)}\Phi_0^{(2)}\cdots\Phi_0^{(i)}\Phi_0^{(i+1)}\cdots\Phi_0^{(n)}\right\rangle = \left\langle\Phi_0^{(1)}\Phi_0^{(2)}\right\rangle\left\langle\Phi_0^{(3)}\Phi_0^{(4)}\right\rangle\cdots\left\langle\Phi_0^{(i)}\Phi_0^{(i+1)}\right\rangle\cdots\left\langle\Phi_0^{(n-1)}\Phi_0^{(n)}\right\rangle$$
$$+ \text{ permutations } \quad (\text{for } n \text{ even}) \tag{7.4}$$

and

$$\left\langle\Phi_0^{(1)}\Phi_0^{(2)}\cdots\Phi_0^{(i)}\Phi_0^{(i+1)}\cdots\Phi_0^{(n)}\right\rangle = 0 \quad (\text{for } n \text{ odd}). \tag{7.5}$$

Therefore C_4 is

$$\langle\Phi_\theta(\mathbf{k}_1,t_1)\Phi_\theta(\mathbf{k}_2,t_2)\Phi_\theta(\mathbf{k}_3,t_3)\Phi_\theta(\mathbf{k}_4,t_4)\rangle = e^{-\frac{i}{2}(\mathbf{k}_3\wedge\mathbf{k}_2+\mathbf{k}_3\wedge\mathbf{k}_1+\mathbf{k}_2\wedge\mathbf{k}_1)}$$
$$\times\left(\left\langle\Phi_0\left(\mathbf{k}_1, t_1-\frac{\vec{\theta}^0\cdot\mathbf{k}_1}{2}\right)\Phi_0\left(\mathbf{k}_2, t_2-\vec{\theta}^0\cdot\mathbf{k}_1-\frac{\vec{\theta}^0\cdot\mathbf{k}_2}{2}\right)\right\rangle\right.$$
$$\times\left\langle\Phi_0\left(\mathbf{k}_3, t_3-\vec{\theta}^0\cdot\mathbf{k}_1-\vec{\theta}^0\cdot\mathbf{k}_2-\frac{\vec{\theta}^0\cdot\mathbf{k}_3}{2}\right)\Phi_0\left(\mathbf{k}_4, t_4-\frac{\vec{\theta}^0\cdot\mathbf{k}_1+\vec{\theta}^0\cdot\mathbf{k}_2+\vec{\theta}^0\cdot\mathbf{k}_3}{2}\right)\right\rangle$$
$$+\left\langle\Phi_0\left(\mathbf{k}_1, t_1-\frac{\vec{\theta}^0\cdot\mathbf{k}_1}{2}\right)\Phi_0\left(\mathbf{k}_3, t_3-\vec{\theta}^0\cdot\mathbf{k}_1-\vec{\theta}^0\cdot\mathbf{k}_2-\frac{\vec{\theta}^0\cdot\mathbf{k}_3}{2}\right)\right\rangle$$
$$\times\left\langle\Phi_0\left(\mathbf{k}_2, t_2-\vec{\theta}^0\cdot\mathbf{k}_1-\frac{\vec{\theta}^0\cdot\mathbf{k}_2}{2}\right)\Phi_0\left(\mathbf{k}_4, t_4-\frac{\vec{\theta}^0\cdot\mathbf{k}_1+\vec{\theta}^0\cdot\mathbf{k}_2+\vec{\theta}^0\cdot\mathbf{k}_3}{2}\right)\right\rangle$$
$$+\left\langle\Phi_0\left(\mathbf{k}_1, t_1-\frac{\vec{\theta}^0\cdot\mathbf{k}_1}{2}\right)\Phi_0\left(\mathbf{k}_4, t_4-\frac{\vec{\theta}^0\cdot\mathbf{k}_1+\vec{\theta}^0\cdot\mathbf{k}_2+\vec{\theta}^0\cdot\mathbf{k}_3}{2}\right)\right\rangle$$
$$\left.\times\left\langle\Phi_0\left(\mathbf{k}_2, t_2-\vec{\theta}^0\cdot\mathbf{k}_1-\frac{\vec{\theta}^0\cdot\mathbf{k}_2}{2}\right)\Phi_0\left(\mathbf{k}_3, t_3-\vec{\theta}^0\cdot\mathbf{k}_1-\vec{\theta}^0\cdot\mathbf{k}_2-\frac{\vec{\theta}^0\cdot\mathbf{k}_3}{2}\right)\right\rangle\right). \tag{7.6}$$

Using spatial translational invariance for each two-point function, we have

$$\langle\Phi_\theta(\mathbf{k}_1,t_1)\Phi_\theta(\mathbf{k}_2,t_2)\Phi_\theta(\mathbf{k}_3,t_3)\Phi_\theta(\mathbf{k}_4,t_4)\rangle = \left[\left\langle\Phi_0\left(\mathbf{k}_1, t_1-\frac{\vec{\theta}^0\cdot\mathbf{k}_1}{2}\right)\Phi_0\left(\mathbf{k}_2, t_2-\frac{\vec{\theta}^0\cdot\mathbf{k}_1}{2}\right)\right\rangle\right.$$
$$\left.\times\left\langle\Phi_0\left(\mathbf{k}_3, t_3-\frac{\vec{\theta}^0\cdot\mathbf{k}_3}{2}\right)\Phi_0\left(\mathbf{k}_4, t_4-\frac{\vec{\theta}^0\cdot\mathbf{k}_3}{2}\right)\right\rangle\right]$$
$$+\mathrm{e}^{-i\mathbf{k}_2\wedge\mathbf{k}_1}\left[\left\langle\Phi_0\left(\mathbf{k}_1, t_1-\frac{\vec{\theta}^0\cdot\mathbf{k}_1}{2}\right)\Phi_0\left(\mathbf{k}_3, t_3-\vec{\theta}^0\cdot\mathbf{k}_2-\frac{\vec{\theta}^0\cdot\mathbf{k}_1}{2}\right)\right\rangle\right.$$

$$\times\left\langle\Phi_0\left(\mathbf{k}_2,t_2-\vec{\theta}^0\cdot\mathbf{k}_1-\frac{\vec{\theta}^0\cdot\mathbf{k}_2}{2}\right)\Phi_0\left(\mathbf{k}_4,t_4-\frac{\vec{\theta}^0\cdot\mathbf{k}_2}{2}\right)\right\rangle\Bigg]$$

$$+\Bigg[\left\langle\Phi_0\left(\mathbf{k}_1,t_1-\frac{\vec{\theta}^0\cdot\mathbf{k}_1}{2}\right)\Phi_0\left(\mathbf{k}_4,t_4-\frac{\vec{\theta}^0\cdot\mathbf{k}_1}{2}\right)\right\rangle$$

$$\times\left\langle\Phi_0\left(\mathbf{k}_2,t_2-\vec{\theta}^0\cdot\mathbf{k}_1-\frac{\vec{\theta}^0\cdot\mathbf{k}_2}{2}\right)\Phi_0\left(\mathbf{k}_3,t_3-\vec{\theta}^0\cdot\mathbf{k}_1-\frac{\vec{\theta}^0\cdot\mathbf{k}_2}{2}\right)\right\rangle\Bigg]. \quad (7.7)$$

Notice that the second term has a non-trivial phase which depends on the spatial momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ and the noncommutative parameter θ. As C_4 cannot be written as sums of products of C_2's in a standard way, we see that the noncommutative probability distribution is non-Gaussian. Also it should be noted that we still cannot achieve Gaussianity of n-point functions even if we modify them by imposing the reality condition as we did for the two-point case.

Non-Gaussianity affects the CMB distribution and also the large scale structure (the large scale distribution of matter in the universe). We have not considered the latter. An upper bound to the amount of non-Gaussianity coming from noncommutativity can be set by extracting the four-point function from the data.

8. Conclusions

In this paper, we have shown that the introduction of spacetime noncommutativity gives rise to nontrivial contributions to the CMB temperature fluctuations. The two-point correlation function in momentum space, called the power spectrum, becomes direction-dependent. Thus spacetime noncommutativity breaks the rotational invariance of the CMB spectrum. That is, CMB radiation becomes statistically anisotropic. This can be measured experimentally to set bounds on the noncommutative parameter. Currently, we [40] are making numerical fits to the available CMB data to put bounds on θ.

We have also shown that the probability distribution governing correlations of fields on the Groenewold-Moyal algebra $\mathcal{A}_\theta$ are non-Gaussian. This affects the correlation functions of temperature fluctuations. By measuring the amount of non-Gaussianity from the four-point correlation function data for temperature fluctuations, we can thus set further limits on θ.

Acknowledgments

We gratefully acknowledge discussions with Cristian Armendariz-Picon and especially with Eric West. This work was partially supported by the US Department of Energy under grant number DE-FG02-85ER40231.

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